Math 2130
HW 3 Solutions

1 $f(x,y) = 5xy^2 - 4x^3y$, $P = (1,2)$

$\vec{u} = \left\langle \frac{5}{13}, \frac{12}{13} \right\rangle$

Note that $\vec{u}$ is a unit vector because

$$|\vec{u}| = \sqrt{\left(\frac{5}{13}\right)^2 + \left(\frac{12}{13}\right)^2} = \sqrt{\frac{5^2+12^2}{13^2}} = \sqrt{1} = 1$$

The gradient of f is

$$\nabla f = \langle f_x, f_y \rangle = \langle 5y^2 - 12x^2y, 10xy - 4x^3 \rangle$$

At $P = (1,2)$, the gradiant is

$$\nabla f = \langle 5 \cdot 2^2 - 12 \cdot 1^2 \cdot 2, 10 \cdot 1 \cdot 2 - 4 \cdot 1^3 \rangle = \langle -4, 16 \rangle$$

Thus, the rate of change of f at P in the direction of $\vec{u}$ is

$$D_{\vec{u}} f(1,2) = \nabla f(1,2) \cdot \vec{u}$$

$$= \langle -4, 16 \rangle \cdot \left\langle \frac{5}{13}, \frac{12}{13} \right\rangle$$

$$= (-4)\left(\frac{5}{13}\right) + (16)\left(\frac{12}{13}\right) = \boxed{\frac{172}{13}} \approx \boxed{13.23}$$

② $f(x,y) = \ln(x^2+y^2)$, $P=(1,2)$, $\vec{v} = \langle -1, 2 \rangle$

$$\nabla f = \langle f_x, f_y \rangle = \left\langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right\rangle$$

$$\nabla f(1,2) = \left\langle \frac{2(1)}{1^2+2^2}, \frac{2(2)}{1^2+2^2} \right\rangle = \left\langle \frac{2}{5}, \frac{4}{5} \right\rangle$$

Note that the length of $\vec{v}$ is

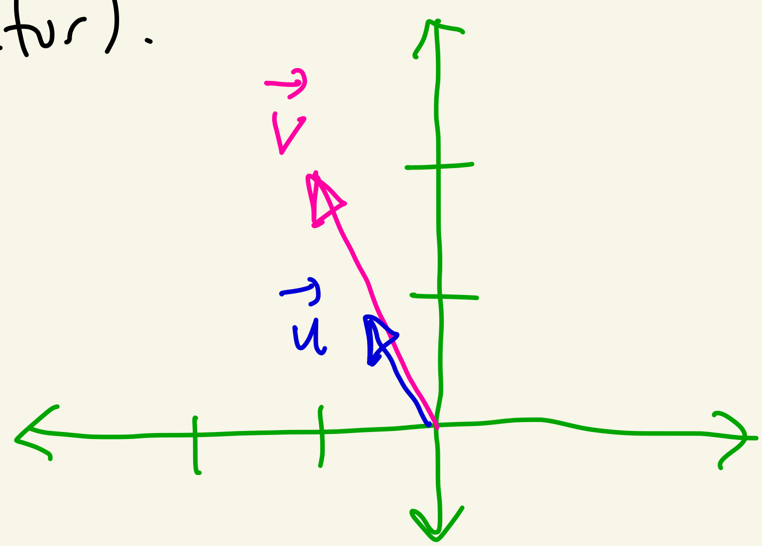
$$|\vec{v}| = \sqrt{(-1)^2+2^2} = \sqrt{5} \approx 2.24$$

this is not 1

So, $\vec{v}$ is not a unit vector.

Let $\vec{u} = \frac{1}{|\vec{v}|} \cdot \vec{v} = \frac{1}{\sqrt{5}} \langle -1, 2 \rangle = \left\langle \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$

$\vec{u}$ points in the direction of $\vec{v}$ but has length 1 (that is, $\vec{u}$ is a unit vector).



We have that the directional derivative of f at $P=(1,2)$ in the direction of $\vec{v}$ is

$$D_{\vec{u}} f(1,2) = \nabla f(1,2) \cdot \vec{u}$$

We need a unit vector here

$$= \left\langle \frac{2}{5}, \frac{4}{5} \right\rangle \cdot \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

$$= \left(\frac{2}{5}\right)\left(-\frac{1}{\sqrt{5}}\right) + \left(\frac{4}{5}\right)\left(\frac{2}{\sqrt{5}}\right)$$

$$= \frac{-2+8}{5\sqrt{5}} = \boxed{\frac{6}{5\sqrt{5}}} = \boxed{\frac{6\sqrt{5}}{25}}$$

$\boxed{3}$ $g(x,y)=\sqrt{xy}$, $P=(2,8)$, $Q=(5,4)$

$$\nabla g = \langle g_x, g_y \rangle = \left\langle \tfrac{1}{2}(xy)^{-1/2}\cdot y,\ \tfrac{1}{2}(xy)^{-1/2}\cdot x \right\rangle$$

$$\nabla g(2,8) = \left\langle \tfrac{1}{2}(2\cdot 8)^{-1/2}\cdot 8,\ \tfrac{1}{2}(2\cdot 8)^{-1/2}\cdot 2 \right\rangle$$

$$= \left\langle \frac{1}{2}\frac{1}{\sqrt{16}}\cdot 8,\ \frac{1}{2}\frac{1}{\sqrt{16}}\cdot 2 \right\rangle$$

$$= \left\langle 1, \frac{1}{4} \right\rangle$$

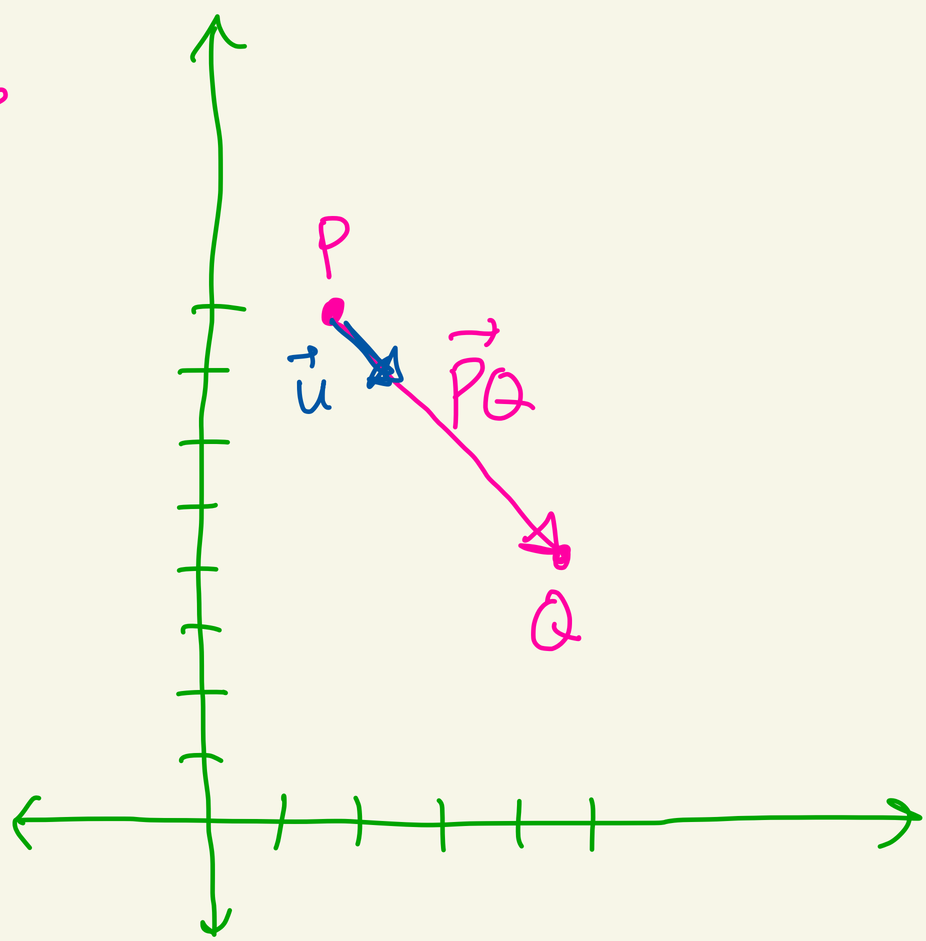
$$\vec{PQ} = \underbrace{\langle 5-2, 4-8 \rangle}_{\text{calculate } Q-P}$$

$$= \langle 3, -4 \rangle$$

We need a unit vector.

$$|\vec{PQ}| = \sqrt{3^2+(-4)^2}$$

$$= \sqrt{9+16} = 5$$



Let

$$\vec{u} = \frac{1}{|\overrightarrow{PQ}|}\overrightarrow{PQ} = \frac{1}{5}\langle 3, -4 \rangle = \left\langle \frac{3}{5}, \frac{-4}{5} \right\rangle$$

see pic above

The directional derivative of g at P in the direction of Q is

$$D_{\vec{u}}\, g(2,8) = \nabla g(2,8) \cdot \vec{u}$$

$$= \left\langle 1, \frac{1}{4} \right\rangle \cdot \left\langle \frac{3}{5}, \frac{-4}{5} \right\rangle$$

$$= (1)\left(\frac{3}{5}\right) + \left(\frac{1}{4}\right)\left(\frac{-4}{5}\right)$$

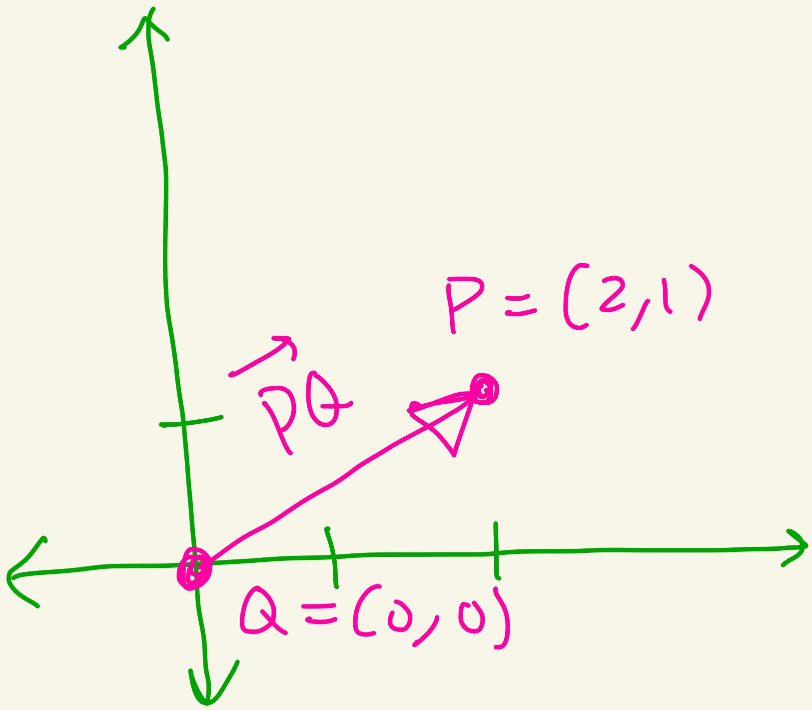
$$= \frac{3}{5} - \frac{1}{5} = \boxed{\frac{2}{5}}$$

$\boxed{④}$

$h(x,y) = x^2 + y^2$

$P = (2,1)$, $Q = (0,0)$

We want the $\vec{PQ}$ direction vector which is



$$\vec{PQ} = \underbrace{\langle 2-0, 1-0 \rangle}_{P-Q \text{ vector}}$$

$$= \langle 2, 1 \rangle$$

The length of $\vec{PQ}$ is

$$|\vec{PQ}| = \sqrt{(-2)^2 + (-1)^2} = \sqrt{5} \approx 2.24$$

So $\vec{PQ}$ is not a unit vector.

We divide it by its length

to make a unit vector:

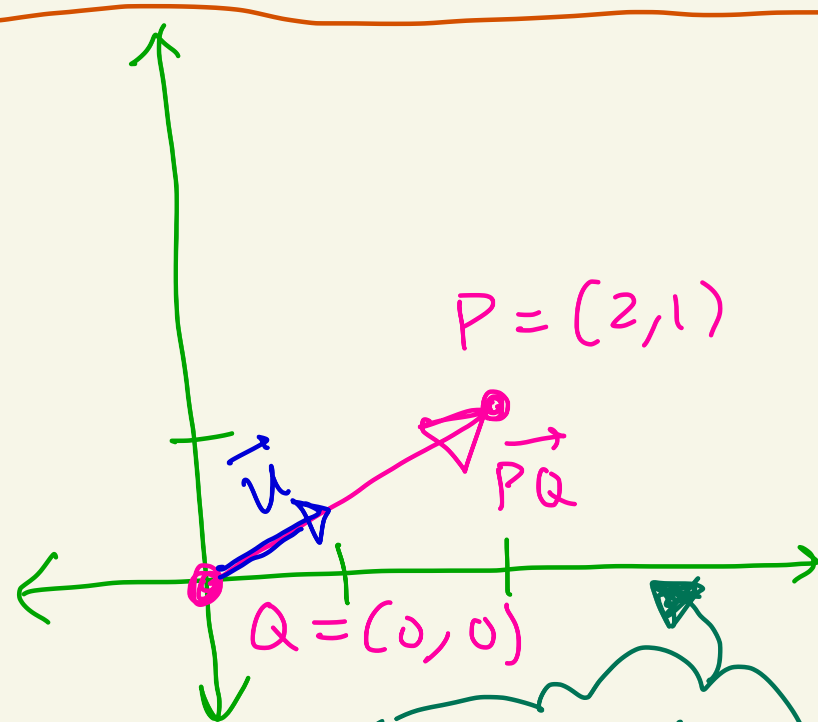
$$\vec{u} = \frac{1}{|\vec{PQ}|}\vec{PQ} = \frac{1}{\sqrt{5}} \cdot \langle 2, 1 \rangle$$

$$= \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

We want

$D_{\vec{u}} h(2,1)$

$= \nabla h(2,1) \cdot \vec{u}$



$\vec{u}$ points in the direction of $\vec{PQ}$ but $\vec{u}$ has length 1

Let's calculate

$\nabla h(2,1)$.

We have

$$\nabla h = \langle h_x, h_y \rangle = \langle 2x, 2y \rangle$$

$$\nabla h(2,1) = \langle 2(2), 2(1) \rangle = \langle 4, 2 \rangle$$

Thus,

$$D_{\vec{u}} h(2,1) = \nabla h(2,1) \cdot \vec{u}$$

$$= \langle 4, 2 \rangle \cdot \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

$$= (4) \cdot \left(\frac{2}{\sqrt{5}}\right) + (2)\left(\frac{1}{\sqrt{5}}\right)$$

$$= \frac{8}{\sqrt{5}} + \frac{2}{\sqrt{5}}$$

$$= \boxed{\frac{10}{\sqrt{5}}} \approx \boxed{22.36}$$

⑤ (a) $f(x,y) = 2x + y$ and $P = (1,0)$

$k = -2$:

$2x + y = -2$

$y = 2x - 2$

$k = 0$:

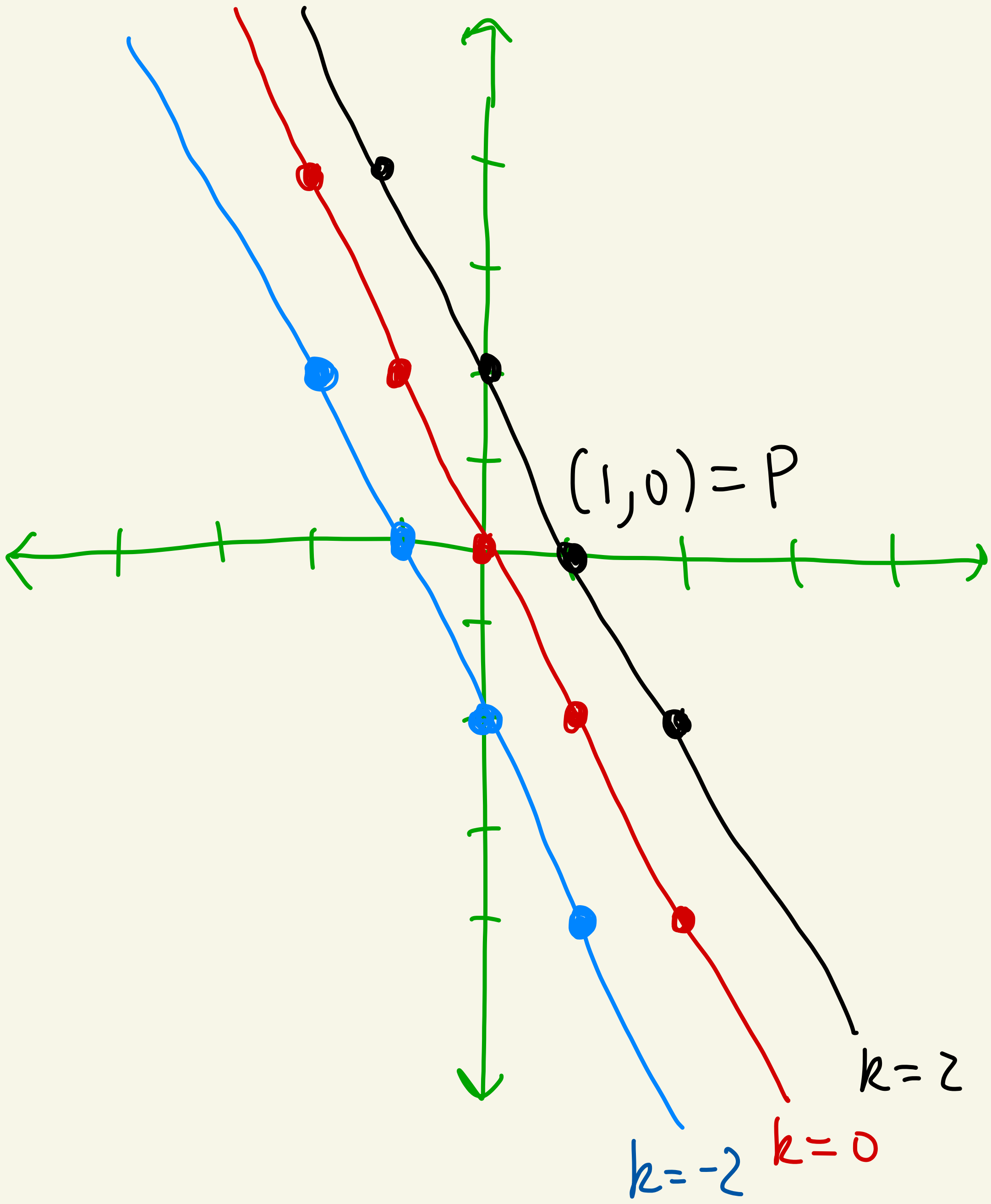
$2x + y = 0$

$y = -2x$

$k = 2$:

$2x + y = 2$

$y = -2x + 2$



P lies on the level curve $y = 2x + 2$ when $k = 2$.

5 (b)

The direction of maximal increase of f at $P=(1,0)$ is $\nabla f(1,0)$.

We have

$$\nabla f = \langle f_x, f_y \rangle = \langle 2, 1 \rangle$$

the gradient is constant everywhere

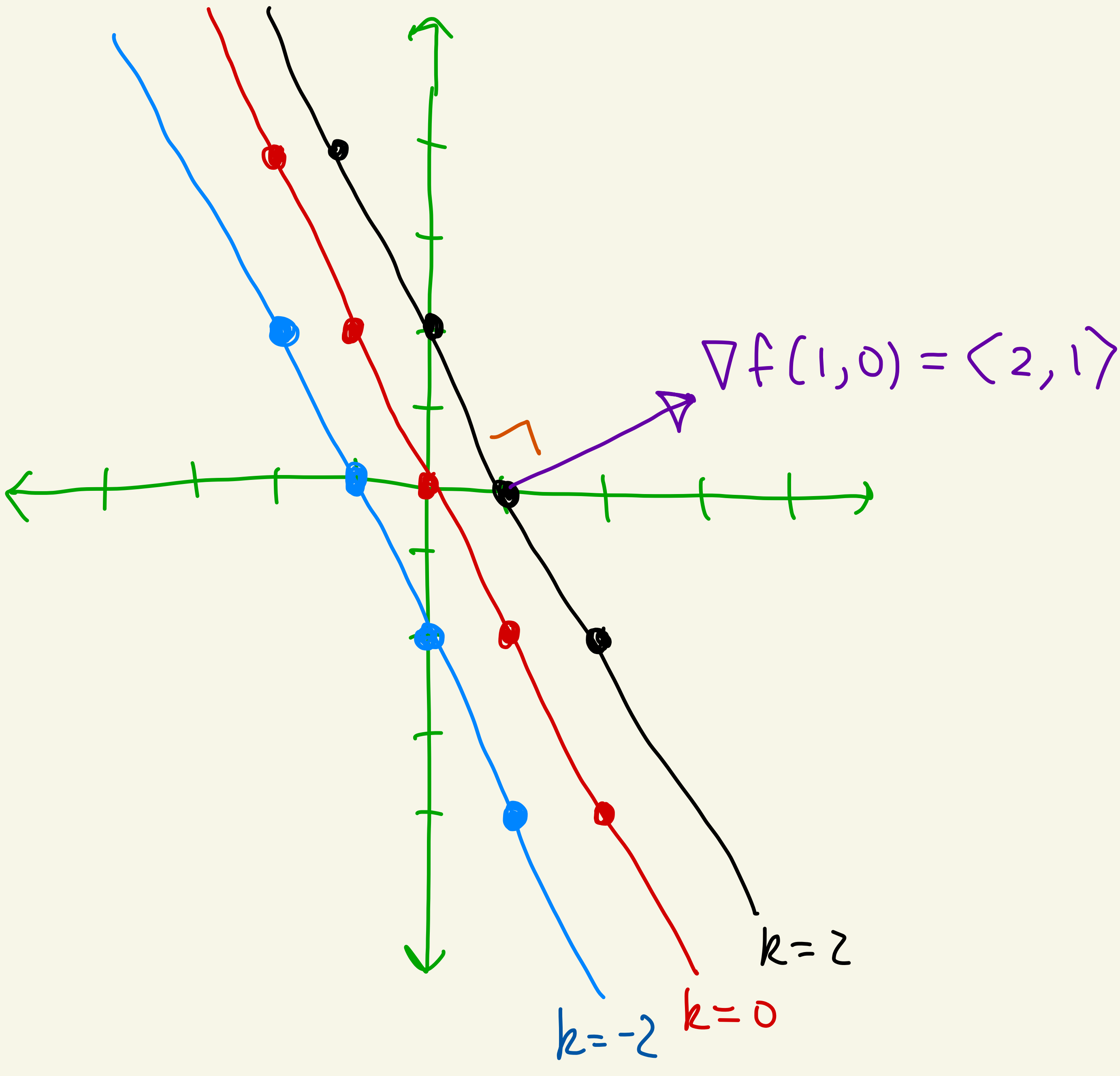
$f = 2x + y$

$$\boxed{\nabla f(1,0) = \langle 2, 1 \rangle}$$

The maximal rate of increase at $P=(0,1)$ is

$$\boxed{|\nabla f(1,0)| = \sqrt{2^2 + 1^2} = \sqrt{5}}$$

5 (c) $\nabla f(1,0) = \langle 2,1 \rangle$



The gradient vector $\nabla f(1,0) = \langle 2,1 \rangle$ is perpendicular to the level curve $k=2$.

6 (a)

$\underline{k=1:}$

$x^2+y^2=1$

$\underline{k=4:}$

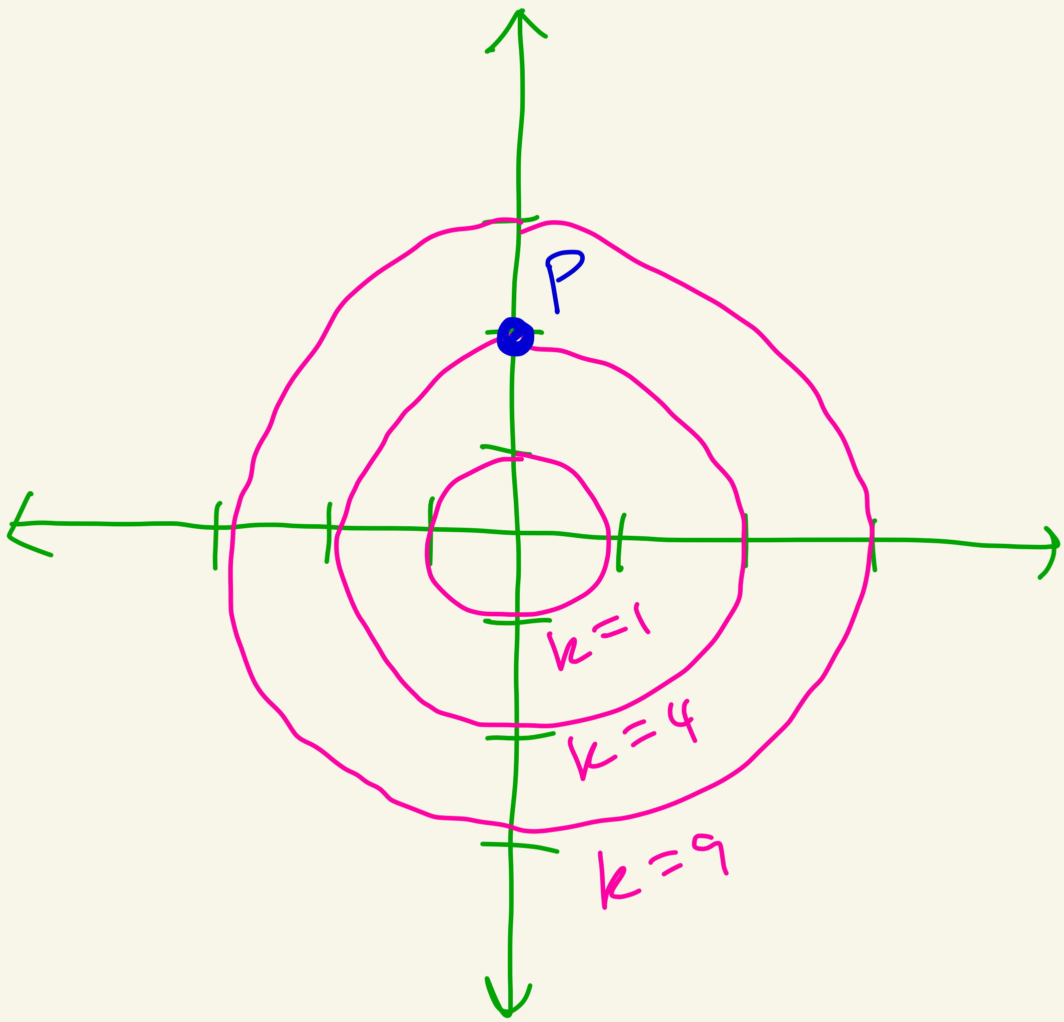
$x^2+y^2=4$

$x^2+y^2=2^2$

$\underline{k=9:}$

$x^2+y^2=9$

$x^2+y^2=3^2$



P lies on the $k=4$ level curve $x^2+y^2=4$.

6 (b) The direction of maximal increase of f at $P=(0,2)$ is the direction of the gradient vector $\nabla f(0,2)$.

We have

$$\nabla f = \langle f_x, f_y \rangle = \langle 2x, 2y \rangle$$

$f = x^2 + y^2$

$$\nabla f(0,2) = \langle 2(0), 2(2) \rangle = \langle 0, 4 \rangle$$

The rate of change in the direction of $\nabla f(0,2)$ is

$$|\nabla f(0,2)| = |\langle 0,4 \rangle|$$

$$= \sqrt{0^2 + 4^2} = 2$$

6(c)

The vector

$\nabla f(0,2) = \langle 0,4 \rangle$

is perpendicular

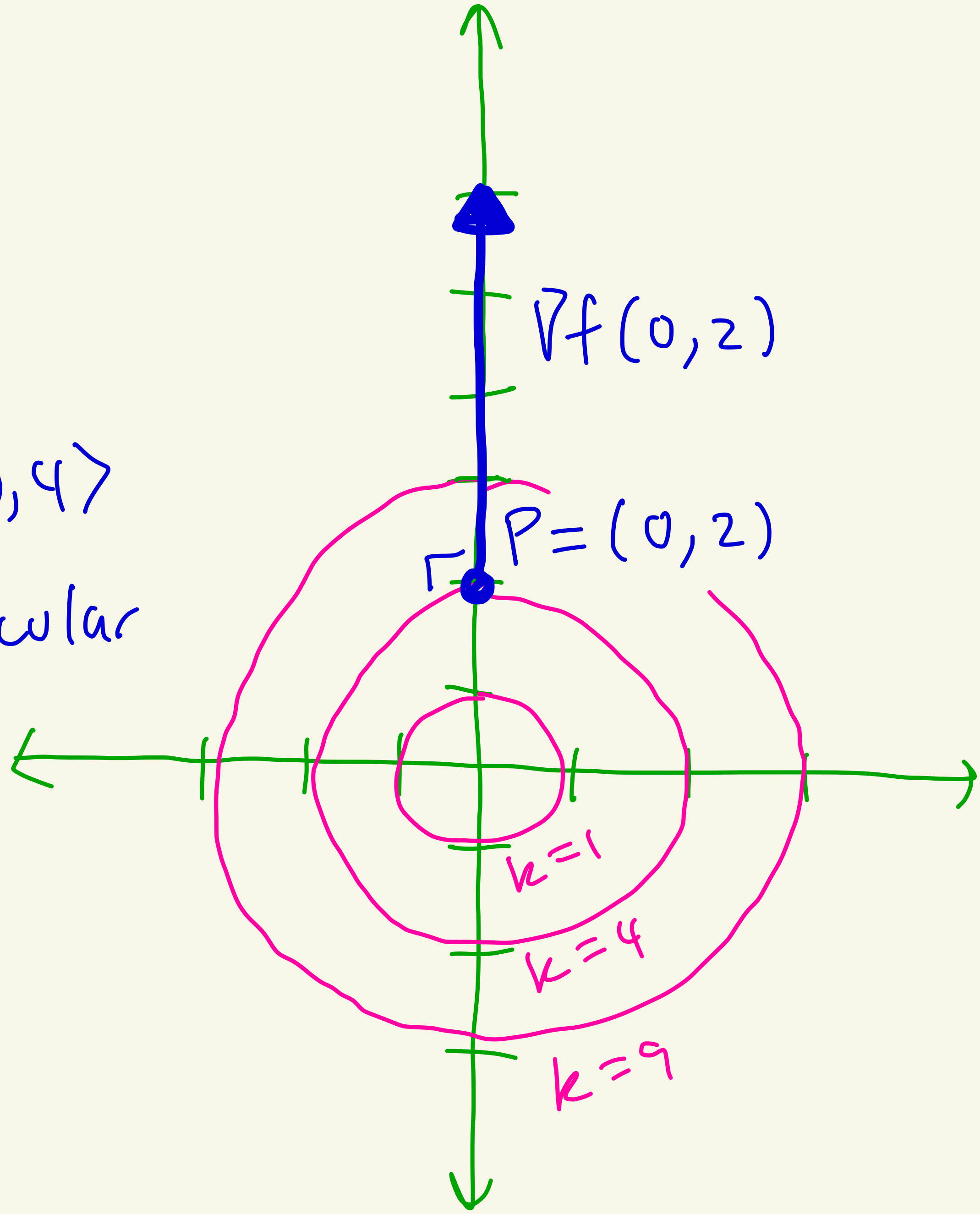
to the

level

curve

$x^2 + y^2 = 4$

$(k=4)$



6 (d)

We want the direction's $\vec{u}$ where $D_{\vec{u}} f(0,2) = 0$.

Let $\vec{u} = \langle a, b \rangle$.
Recall that $\nabla f(0,2) = \langle 0, 4 \rangle$

Then

$$D_{\vec{u}} f(0,2) = 0$$

$$\nabla f(0,2) \cdot \vec{u} = 0$$

$$\langle 0, 4 \rangle \cdot \langle a, b \rangle = 0$$

$$0(a) + 4(b) = 0$$

$$4b = 0$$

$$b = 0$$

Thus $b = 0$.

So, $\vec{u} = \langle a, b \rangle = \langle a, 0 \rangle$.

We need $\vec{u}$ to be a unit vector so we need $|\vec{u}| = 1$.

We need

$$\underbrace{\sqrt{a^2 + 0^2}}_{|\vec{u}|} = 1$$

We get

$$\sqrt{a^2} = 1$$

square both sides

$$a^2 = 1$$

$$a = \pm\sqrt{1} = \pm 1.$$

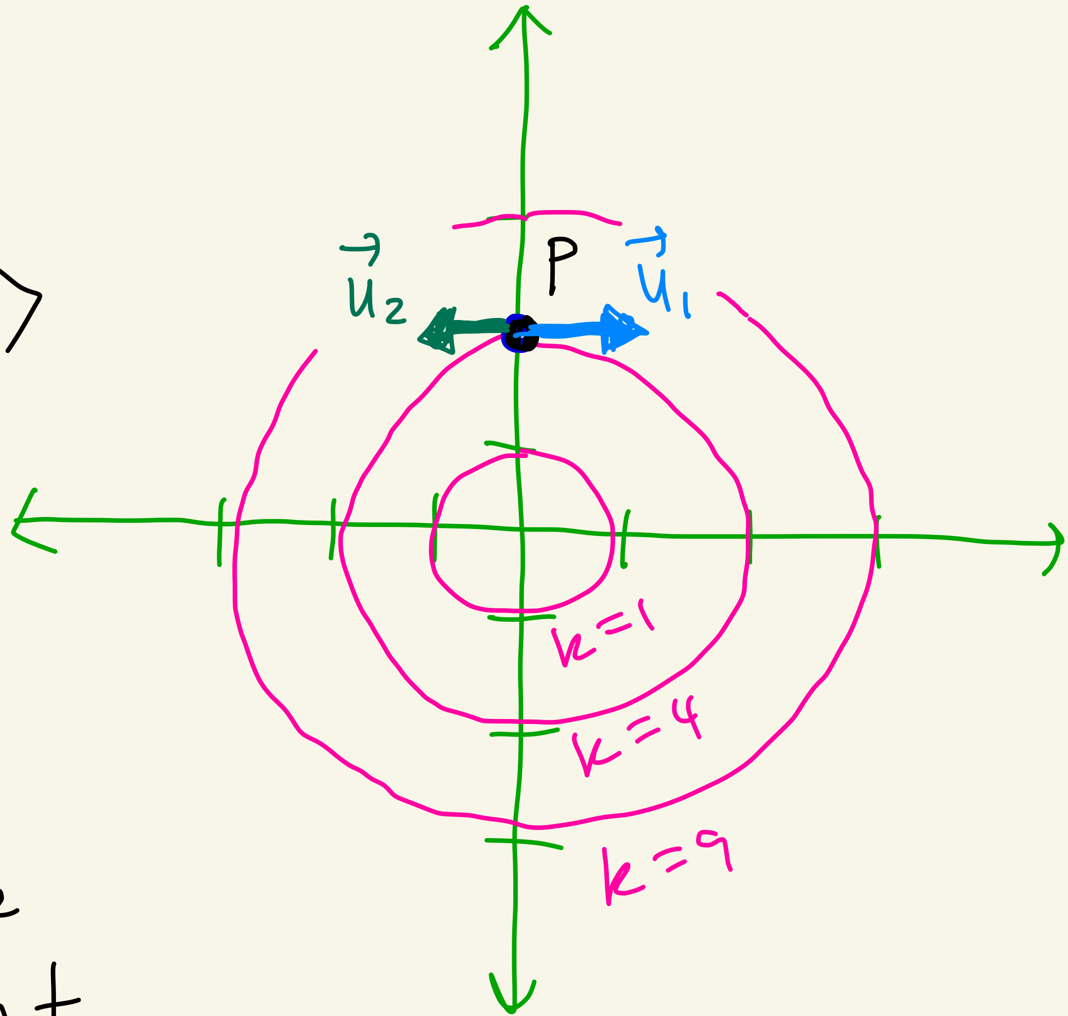
So, $a = \pm 1$.

Thus, $\boxed{\vec{u} = \langle 0, 1 \rangle}$ or $\boxed{\vec{u} = \langle 0, -1 \rangle}$

6 (e)

$\vec{u}_1 = \langle 0, 1 \rangle$

$\vec{u}_2 = \langle 0, -1 \rangle$



$\vec{u}_1$ and $\vec{u}_2$ are tangent or parallel to the curve $x^2 + y^2 = 4$.